THE COHOMOLOGY RING OF THE GKM GRAPH OF A FLAG MANIFOLD (Transformation Groups and Surgery Theory)

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数理解析研究所講究録
第 1732 巻 2011 年 86-91

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1. Introduction

Let T be a torus of dimension n and M a closed smooth T-manifold. The equivariant cohomology of M, denoted $H_T^*(M)$, contains a lot of geometrial information on M. Moreover it is often easier to compute $H_T^*(M)$ than $H^*(M)$ by virtue of the Localization Theorem which implies that the restriction map

$$\iota^*: H_T^*(M) \to H_T^*(M^T) \tag{1.1}$$

to the T-fixed point set M^T is often injective, in fact, this is the case when $H^{odd}(M) = 0$. When M^T is isolated, $H_T^*(M^T) = \oplus_{p\in M^T} H_T^*(p)$ and hence $H_T^*(M^T)$ is a direct sum of copies of a polynomial ring in n variables because $H_T^*(p) = H^*(BT)$.

Therefore we are in a nice situation when $H^{odd}(M) = 0$ and M^T is isolated. Goresky-Kottwitz-MacPherson [2] (see also [3, Chapter 11]) found that under the further condition that the weights at a tangential T-module are pairwise linearly independent at each $p \in M^T$, the image of ι^* in (1.1) above is determined by the fixed point sets of codimension one subtori of T when $\mathbb{Q}$ is tensored in cohomology. Their result motivated Guillemin-Zara [4] to associate a labeled graph $\mathcal{G}_M$ with M and define the "cohomology" ring $\mathcal{H}^*(\mathcal{G}_M)$ of $\mathcal{G}_M$, which is a subring of $\oplus_{p\in M^T} H^*(BT)$. Then the result of Goresky-Kottwitz-MacPherson can be stated that $H_T^*(M)\otimes\mathbb{Q}$ is isomorphic to $\mathcal{H}^*(\mathcal{G}_M)\otimes\mathbb{Q}$ as graded rings when M satisfies the conditions mentioned above.

The result of Goresky-Kottwitz-MacPherson can be applied to many important T-manifolds M such as flag manifolds and compact smooth toric varieties etc. When M is such a nice manifold, $H_T^*(M)$ is often known to be isomorphic to $\mathcal{H}^*(\mathcal{G}_M)$ without tensoring with $\mathbb{Q}$ (see [1], [5], [6] for example). We determine the ring structure of $\mathcal{H}^*(\mathcal{G}_M)$ or $\mathcal{H}^*(\mathcal{G}_M)\otimes\mathbb{Z}[\frac{1}{2}]$ when M is a flag manifold of classical type directly without using the fact that $H_T^*(M)$ is isomorphic to $\mathcal{H}^*(\mathcal{G}_M)$ ([7]). In my talk, I introduced the result when M is a flag manifold of type A. This is a joint work with Hiroaki Ishida and Mikiya Masuda and the details can be found in [7].

2. Labeled graph and its cohomology for type A_{n-1}

Let $\{t_i\}_{i=1}^n$ be a basis of $H^2(BT)$, so that $H^*(BT)$ can be identified with a polynomial ring $\mathbb{Z}[t_1, t_2, \dots, t_n]$. We take an inner product on $H^2(BT)$ such that the basis $\{t_i\}$ is orthonormal. Then

$$\Phi(A_{n-1}) := \{\pm(t_i - t_j) \mid 1 \le i < j \le n\} \tag{2.1}$$

is a root system of type A_{n-1}.

Definition. The labeled graph associated with $\Phi(A_{n-1})$, denoted $\mathcal{A}_n$, is a graph with labeling ℓ defined as follows.

- The vertex set of $\mathcal{A}_n$ is the permutation group S_n on {1,2,...,n}.
- Two vertices w, w' in $\mathcal{A}_n$ are connected by an edge $e_{w,w'}$ if and only if there is a transposition $(i, j) \in S_n$ such that $w' = w(i, j)$, in other words,

  $$w'(i) = w(j), \quad w'(j) = w(i) \quad and \quad w'(r) = w(r) \quad for \quad r \neq i,\ j.$$
- The edge $e_{w,w'}$ is labeled by $\ell(e_{w,w'}) := t_{w(i)} - t_{w'(i)}$.

Definition. The cohomology ring of $\mathcal{A}_n$, denoted $\mathcal{H}^*(\mathcal{A}_n)$, is defined to be the subring of $\mathrm{Map}(V(\mathcal{A}_n), H^*(BT)) = \bigoplus_{v \in V(\mathcal{A}_n)} H^*(BT)$, where $V(\mathcal{A}_n)$ denotes the set of vertices of $\mathcal{A}_n$, *i.e.* $V(\mathcal{A}_n) = S_n$, satisfying the following condition:

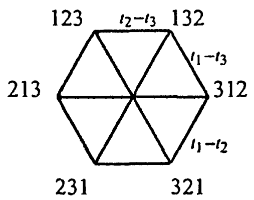
> $f \in \mathrm{Map}(V(\mathcal{A}_n), H^*(BT))$ is an element of $\mathcal{H}^*(\mathcal{A}_n)$ if and only if $f(v) - f(v')$ is divisible by $\ell(e)$ in $H^*(BT)$ whenever the vertices v and v' are connected by an edge e in $\mathcal{A}_n$.

For each $i = 1, \dots, n$, we define elements τ_i, t_i of $\mathrm{Map}(V(\mathcal{A}_n), H^*(BT))$ by

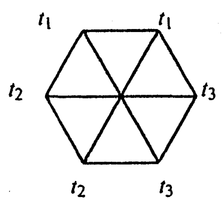
$$\tau_i(w) := t_{w(i)}, \quad t_i(w) := t_i \quad \text{for } w \in S_n. \tag{2.2}$$

In fact, both τ_i and t_i are elements of $\mathcal{H}^2(\mathcal{A}_n)$.

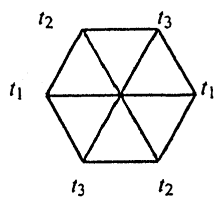
Example. The case $n = 3$. The root system $\Phi(A_2)$ is $\{\pm(t_i - t_j) | 1 \le i < j \le 3\}$. The labeled graph $\mathcal{A}_3$ and τ_i for $i = 1, 2, 3$ are as follows.



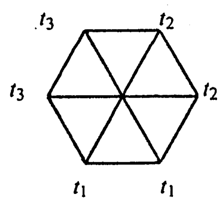
The labeled graph $\mathcal{A}_3$



τ_1



τ_2



τ_3

Theorem 2.1. *Let $\mathcal{A}_n$ be the labeled graph associated with the root system $\Phi(A_{n-1})$ of type A_{n-1} in (2.1). Then*

$$\mathcal{H}^*(\mathcal{A}_n) = \mathbb{Z}[\tau_1, \cdots, \tau_n, t_1, \cdots, t_n]/(e_i(\tau) - e_i(t) \mid i = 1, \cdots, n),$$

where $e_i(\tau)$ (resp. $e_i(t)$) is the i^{th} elementary symmetric polynomial in $\tau_1, \cdots, \tau_n$ (resp. $t_1, \cdots, t_n$).

To prove this theorem, we need the following two lemmas.

Lemma 2.2. *$\mathcal{H}^*(\mathcal{A}_n)$ is generated by $\tau_1, \cdots, \tau_n, t_1, \cdots, t_n$ as a ring.*

Proof. We shall prove the lemma by induction on n. When $n = 1$, $\mathcal{H}^*(\mathcal{A}_1)$ is generated by t_1 since $\mathcal{A}_1$ is a point; so the lemma holds.

Suppose that the lemma holds for $n-1$. Then it suffices to show that any homogenous element f of $\mathcal{H}^*(\mathcal{A}_n)$, say of degree $2k$, can be expressed as a polynomial in τ_i's and t_i's. For each $i = 1, \ldots, n$, we set

$$V_i := \{w \in S_n \mid w(i) = n\}$$

and consider the labeled full subgraph $\mathcal{L}_i$ of $\mathcal{A}_n$ with V_i as the vertex set. Note that $\mathcal{L}_i$ can naturally be identified with $\mathcal{A}_{n-1}$ for any i.

Let

$$1 \le q \le \min\{k+1, n\} \tag{2.3}$$

and assume that

$$f(v) = 0 \quad \text{for any } v \in V_i \text{ whenever } i < q. \tag{2.4}$$

A vertex w in V_q is connected by an edge in $\mathcal{A}_n$ to a vertex v in V_i if and only if $v = w(i, q)$. In this case $f(w) - f(v)$ is divisible by $t_{w(i)} - t_{w(q)} = t_{w(i)} - t_n$ and $f(v) = 0$ whenever $i < q$ by (2.4), so $f(w)$ is divisible by $t_{w(i)} - t_n$ for $i < q$. Thus, for each $w \in V_q$, there is an element $g^q(w) \in \mathbb{Z}[t_1, \cdots, t_n]$ such that

$$f(w) = (t_{w(1)} - t_n)(t_{w(2)} - t_n) \ldots (t_{w(q-1)} - t_n) g^q(w) \tag{2.5}$$

where $g^q(w)$ is homogeneous and of degree $2(k+1-q)$ because $f(w)$ is homogenous and of degree $2k$.

One expresses

$$g^q(w) = \sum_{r=0}^{k+1-q} g^q_r(w) t_n^r \tag{2.6}$$

with homogenous polynomials $g^q_r(w)$ of degree $2(k+1-q-r)$ in $\mathbb{Z}[t_1, \cdots, t_{n-1}]$. Then there is a polynomial G^q_r in τ_i's (except τ_q) and t_i's (except t_n) such that $G^q_r(w) = g^q_r(w)$ for any $w \in V_q$, because g^q_r restricted to $\mathcal{L}_q$ is an element of $\mathcal{H}^*(\mathcal{L}_q) = \mathcal{H}^*(\mathcal{A}_{n-1})$.

Since $\tau_i(w) = t_{w(i)}$ and $w(i) = n$ for $w \in V_i$, we have

$$(2.7) \qquad \prod_{i=1}^{q-1}(\tau_i - t_n)(w) = 0 \quad \text{for any } w \in V_i \text{ whenever } i < q.$$

Therefore, it follows from (2.5), (2.6), the Claim above and (2.7) that putting $G^q = \sum_{r=0}^{k+1-q} G^q_r t^r_n$, we have

$$\begin{aligned}(f - G^q \prod_{i=1}^{q-1}(\tau_i - t_n))(w) &= f(w) - g^q(w)\prod_{i=1}^{q-1}(t_{w(i)} - t_n) \\ &= 0 \qquad \text{for any } w \in V_i \text{ whenever } i \le q.\end{aligned}$$

Therefore, subtracting the polynomial $G^q \prod_{i=1}^{q-1}(\tau_i - t_n)$ from f, we may assume that

$$f(v) = 0 \quad \text{for any } v \in V_i \text{ whenever } i < q + 1.$$

The above argument implies that f finally takes zero on all vertices of $\mathcal{A}_n$ (which means $f = 0$) by subtracting a polynomial in τ_i's and t_i's, and this completes the induction step. □

We abbreviate the polynomial ring $\mathbb{Z}[\tau_1, \cdots, \tau_n, t_1, \cdots, t_n]$ as $\mathbb{Z}[\tau, t]$. The canonical map $\mathbb{Z}[\tau, t] \to \mathcal{H}^*(\mathcal{A}_n)$ is a grade preserving homomorphism which is surjective by Lemma 2.2. Let $e_i(\tau)$ (resp. $e_i(t)$) denote the i^{th} elementary symmetric polynomial in $\tau_1, \cdots, \tau_n$ (resp. $t_1, \cdots, t_n$). It easily follows from (2.2) that $e_i(\tau) = e_i(t)$ for $i = 1, \cdots, n$. Therefore the canonical map above induces a grade preserving epimorphism

$$(2.8) \qquad \mathbb{Z}[\tau, t]/(e_1(\tau) - e_1(t), \cdots, e_n(\tau) - e_n(t)) \to \mathcal{H}^*(\mathcal{A}_n).$$

Remember that the Hilbert series of a graded ring $A^* = \oplus_{j=0}^{\infty} A^j$, where A^j is the degree j part of A^* and of finite rank over $\mathbb{Z}$, is a formal power series defined by

$$F(A^*, s) := \sum_{j=0}^{\infty}(\operatorname{rank}_{\mathbb{Z}} A^j)s^j.$$

In order to prove that the epimorphism in (2.8) is an isomorphism, it suffices to verify the following lemma because the modules in (2.8) are both torsion free.

Lemma 2.3. *The Hilbert series of the both sides at* (2.8) *coincide, in fact, they are given by* $\frac{1}{(1-s^2)^{2n}} \prod_{i=1}^{n}(1 - s^{2i})$.

Proof. (1) Calculation of LHS at (2.8). Let $e_i := e_i(\tau) - e_i(t)$. It follows from the exact sequence

$$0 \to (e_1, \cdots, e_n) \to \mathbb{Z}[\tau, t] \to \mathbb{Z}[\tau, t]/(e_1, \cdots, e_n) \to 0$$

that we have

$$(2.9)\qquad F(\mathbb{Z}[\tau,t]/(e_1,\cdots,e_n),s) = F(\mathbb{Z}[\tau,t],s) - F((e_1,\cdots,e_n),s).$$

Here, since $\deg\tau_i = \deg t_i = 2$, we have

$$(2.10)\qquad F(\mathbb{Z}[\tau,t],s) = \frac{1}{(1-s^2)^{2n}}$$

as easily checked; so it suffices to calculate $F((e_1,\cdots,e_n),s)$.

For $I \subset [n]$ we set $e_I := \prod_{i\in I} e_i$. Then it follows from the Inclusion-Exclusion principle that

$$(2.11)\qquad F((e_1,\cdots,e_n),s) = \sum_{\emptyset\neq I\subset[n]} (-1)^{|I|-1} F((e_I),s)$$

and since $F((e_I),s) = s^{\deg e_I}/(1-s^2)^{2n}$ and $\deg e_I = \sum_{i\in I} 2i$, it follows from (2.11) that

$$(2.12)\qquad F((e_1,\cdots,e_n),s) = \sum_{\phi\neq I\subset[n]} (-1)^{|I|-1} \frac{s^{\sum_{i\in I} 2i}}{(1-s^2)^{2n}}.$$

Therefore it follows from (2.9), (2.10) and (2.12) that

$$\begin{aligned} F(\mathbb{Z}[\tau,t]/(e_1,\cdots,e_n),s) &= \frac{1}{(1-s^2)^{2n}} - \sum_{\phi\neq I\subset[n]} (-1)^{|I|-1} \frac{s^{\sum_{i\in I} 2i}}{(1-s^2)^{2n}} \\ (2.13)\qquad &= \frac{1}{(1-s^2)^{2n}} \sum_{I\subset[n]} (-1)^{|I|} s^{\sum_{i\in I} 2i} \\ &= \frac{1}{(1-s^2)^{2n}} \prod_{i=1}^{n} (1-s^{2i}). \end{aligned}$$

(2) Calculation of RHS at (2.8). Let $d_n(k) := \operatorname{rank}_{\mathbb{Z}} \mathcal{H}^{2k}(\mathcal{A}_n)$. Then

$$(2.14)\qquad F(\mathcal{H}^*(\mathcal{A}_n),s) = \sum_{k=0}^{\infty} d_n(k) s^{2k}.$$

Recall the argument in the proof of Lemma 2.2. Since g_r^q in (2.6) belongs to $\mathcal{H}^{2(k+1-q-r)}(\mathcal{L}_q) = \mathcal{H}^{2(k+1-q-r)}(\mathcal{A}_{n-1})$ as shown in the Claim there, the rank of the module consisting of those g^q in (2.5) and (2.6) is given by

$$\sum_{r=0}^{k+1-q} d_{n-1}(k+1-q-r) = \sum_{r=0}^{k+1-q} d_{n-1}(r).$$

Therefore, noting (2.3), we see that the argument in the proof of Lemma 2.2 implies

$$d_n(k) = \sum_{q=1}^{\min\{k+1,n\}} \sum_{r=0}^{k+1-q} d_{n-1}(r),$$

in other words, if we set $d_{n-1}(j) = 0$ for $j < 0$, then
(2.15)

$$d_n(k) = \begin{cases} \sum_{i=1}^{n} i d_{n-1}(k+1-i) & \text{if } k \le n-1, \\ \sum_{i=1}^{n} i d_{n-1}(k+1-i) + n \sum_{i=n+1}^{k+1} d_{n-1}(k+1-i) & \text{if } k \ge n. \end{cases}$$

We shall abbreviate $F(\mathcal{H}^*(\mathcal{A}_n), s)$ as $F_n(s)$. Then, plugging (2.15) in (2.14), we obtain

$$\begin{aligned} F_n(s) &= \sum_{k=0}^{\infty} \big(d_{n-1}(k) + 2d_{n-1}(k-1) + \cdots + nd_{n-1}(k+1-n)\big)s^{2k} \\ &\quad + n \sum_{k=n}^{\infty} \big(d_{n-1}(k-n) + \cdots + d_{n-1}(1) + d_{n-1}(0)\big)s^{2k} \\ &= F_{n-1}(s) + 2s^2 F_{n-1}(s) + \cdots + ns^{2n-2} F_{n-1}(s) \\ &\quad + n\Big(d_{n-1}(0)s^{2n}\frac{1}{1-s^2} + d_{n-1}(1)s^{2n+2}\frac{1}{1-s^2} + \cdots\Big) \\ &= F_{n-1}(s)\big(1 + 2s^2 + \cdots + ns^{2n-2}\big) + n\frac{s^{2n}}{1-s^2}F_{n-1}(s) \\ &= \frac{1-s^{2n}}{1-s^2}F_{n-1}(s). \end{aligned}$$

On the other hand, $F_1(s) = 1/(1-s^2)$ since $\mathcal{H}^*(\mathcal{A}_1) = \mathbb{Z}[t_1]$. It follows that

$$F_n(s) = \frac{1}{(1-s^2)^{2n}} \prod_{i=1}^{n} (1 - s^{2i}).$$

This together with (2.13) proves the lemma. □

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